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Zero-Point Energy of Quantum Fields in a Schwarzschild Geometry

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The effective Lagrangian and the zero-point (or Casimir) energy is calculated from the zeta-function which is obtained by the heat kernel method using the expansion of (Bormann and Antonsen, 1995). Calculated this way this unavoidable energy contribution is automatically regularised and ready for further investigation. Interesting observations include a large energy contribution (from scalar field and fermionic zero-point fluctuations) that is non-zero as the mass goes to zero, perhaps indicating a topological origin. Also, plots of the contribution of gauge boson fields to the zero-point energy, as a function of radial distance (gravitational field strength) and the size of the gauge boson coupling (gauge field strength) shows great variation, notably the occurrence of 'resonances.'

KEY WORDS: zero-point energy in curved space scalars; fermions; gauge fields Schwarzschild geometry.

1. INTRODUCTION

Under investigation is the influence of the gravitational field, in this case given by the Schwarzschild metric which line element is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (1)$$

where M is the mass of the (classical) object generating the Schwarzschild geometry, on the quantum fluctuations and conversely, the influence of the quantum fluctuations on the gravitational field (back-reaction). The theoretical setting is that of quantum matter fields in a gravitational background field (first quantisation).

The Casimir effect is of course unavoidable so the simplest possible action is $S_{\text{Einstein}} + S_{\text{matter}}$, where 'matter' comprises all known quantum fields (except

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gravitation), which leads to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \langle T_{\mu\nu}\rangle^{\text{Casimir}}_{\text{zero}} + T^{\text{classical}}_{\mu\nu} \tag{2}$$

where units are chosen such that $\kappa = \hbar = c = 1$.

In a Schwarzschild space–time $T^{\text{classical}}_{\mu\nu}$ vanishes everywhere except at the origin and we are left with determining $\langle T_{\mu\nu}\rangle^{\text{Casimir}}$, the zero-point energy of the matter fields.

The components of the Casimir energy–momentum tensor (for a given matter field in a given gravitational background) can be determined from the effective action by taking the functional derivative with respect to the inverse metric

$$\langle T_{\mu\nu}\rangle^{\text{Casimir}} = \frac{\delta\Gamma_{\text{eff}}}{\delta g^{\mu\nu}} \tag{3}$$

The effective action (and the effective Lagrangian, L_{eff}) in turn is determined from a path integral. If the functional integral is Gaussian it can be performed, determining the determinant of the operator, A, related to the quantum field.[4]

$$Z = e^{\Gamma_{\text{eff}}} = e^{\int\sqrt{-g}d^4x\mathcal{L}_{\text{eff}}} \equiv \int e^{iS}\mathcal{D}\phi = \int e^{i\int\sqrt{-g}d^4x\phi A\phi}\,\mathcal{D}\phi = (\det(A))^p \tag{4}$$

where g is the metric determinant and where ϕ is a generic matter field and where $p = -\frac{1}{2}$ for a real scalar field, $p = 1$ for a spin 1/2 fermion field, i.e. a complex Grassman field ($p = \frac{1}{2}$ for a real Grassman field) and $p = -1$ for a massless spin 1 gauge boson field. The determinant, in turn, can be determined from the zeta-function, $\zeta_A(s)$ by the relation

$$-\ln\det(A) = \frac{d\zeta_A}{ds}|_{s=0}, \quad \zeta_A(S) \equiv \frac{1}{\Gamma(s)}\int d\sigma\,\sigma^{s-1}\int G_A(x,x;\sigma)\sqrt{g}d^4x \tag{5}$$

where the heat kernel, $G_A(x,\tilde{x};\sigma)$, is the function solving the (heat kernel) equation

$$AG_A(x,\tilde{x};\sigma) = -\frac{\partial}{\partial\sigma}G_A(x,\tilde{x};\sigma) \tag{6}$$

subject to the boundary condition $G_A(x,\tilde{x};0) = \delta(x-\tilde{x})$. In case that the heat kernel is spinor or tensor valued (as it is in the spin 1/2 and spin 1 cases, respectively) one also has to take the trace when determining the effective action or Lagrangian. The heat kernel is then determined by the method developed in (Bormann and Antonsen, 1995) the steps of which we will go through explicitly (for the case of a Schwarzschild metric) in sections 1–3. In section 4 a conclusion and outlook is given.

[4] The relations presented in this introduction are discussed in detail in for instance (Bormann and Antonsen, 1995).

2. THE ZERO-POINT CONTRIBUTION TO THE EFFECTIVE ACTION OF A SCALAR FIELD FOR THE SCHWARZSCHILD METRIC

The scalar field operator is given by

$$\begin{aligned}\Box + m^2 + \xi R &= \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu) + m^2 + \xi R \\ &= g^{\mu\nu}\partial_u\partial_\nu + \frac{1}{\sqrt{-g}}(\partial_\mu\sqrt{-g})g^{\mu\nu}\partial_\nu + (\partial_\mu g^{\mu\nu})\partial_\nu + m^2 + \xi R \\ &= g^{\mu\nu}\partial_\mu\partial_\nu + \left[\partial_\mu g^{\mu\nu} + g^{\mu\nu}\Gamma^\alpha_{\mu\alpha}\right]\partial_\nu + m^2 + \xi R \end{aligned} \tag{7}$$

where $\Box$ is the curved space d'Alembertian, $\Gamma^\alpha_{\mu\alpha}$ a contracted Christoffel symbol and R the curvature scalar. The covariant derivative of the metric vanishes;

$$D_\alpha g^{\mu\nu} = \partial_\alpha g^{\partial\nu} + \Gamma^\mu_{\delta\alpha} g^{\delta\nu} + \Gamma^\nu_{\delta\alpha} g^{\mu\delta} \equiv 0 \tag{8}$$

Contracting α and μ, one can use this equation to simplify the scalar field operator, obtaining

$$\Box + m^2 + \xi R = g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\alpha}\Gamma^\nu_{\alpha\mu}\partial_\nu + m^2 + \xi R \tag{9}$$

Written explicitly, for the case of a Schwarzschild metric, the operator becomes

$$\begin{aligned}\Box + m^2 + \xi R = {} & \frac{1}{1-\frac{2M}{r}}\partial_t^2 - \left(1-\frac{2M}{r}\right)\partial_r^2 - \frac{2r-2M}{r^2}\partial_r - \frac{1}{r^2}\partial_\theta^2 - \frac{\cot(\theta)}{r^2}\partial_\theta \\ & - \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 + m^2 + \xi R \\ \equiv {} & \frac{1}{1-\frac{2M}{r}}\partial_t^2 - \left(1-\frac{2M}{r}\right)\partial_r^2 - \frac{2r-2M}{r^2}\partial_r - \frac{L^2}{r^2} + m^2 + \xi R \end{aligned} \tag{10}$$

where the angular momentum operator squared, L^2, has been introduced. Now change to new coordinates defined by

$$\begin{aligned} t' &= \sqrt{1-\frac{2M}{r}}t \\ r' &= \sqrt{r^2-2Mr} + M\ln(\sqrt{r^2-2Mr} + r - M) \end{aligned} \tag{11}$$

giving the derivatives

$$\partial_t^2 = \frac{\partial^2 t'}{\partial t^2}\partial_{t'} + \left(\frac{\partial t'}{\partial t}\right)^2\partial_{t'}^2 = \left(1-\frac{2M}{r}\right)\partial_{t'}^2 \quad \partial_r = \frac{\partial^{r'}}{\partial r}\partial_{r'} = \frac{1}{\sqrt{1-\frac{2M}{r}}}\partial_{r'}$$

$$\partial_r^2 = \frac{\partial^2 r'}{\partial r^2}\partial_{r'} + \left(\frac{\partial r'}{\partial r}\right)^2 \partial_{r'}^2 = \partial_r\left(\frac{1}{\sqrt{1-\frac{2M}{r}}}\right)\partial_{r'} + \frac{1}{1-\frac{2M}{r}}\partial_{r'}^2 \tag{12}$$

so that the scalar field operator becomes

$$\Box + m^2 + \xi R = \partial_{t'}^2 - \partial_{r'}^2 + \frac{-2r+3M}{r^2\sqrt{1-\frac{2M}{r^2}}}\partial_{r'} - \frac{L^2}{r^2} + m^2 + \xi R \tag{13}$$

Write the corresponding heat kernel equation as

$$\left[\partial_{t'}^2 - \partial_{r'}^2 + g(r(r'))\partial_{r'} - \frac{L^2}{r^2} + m^2 + \xi R\right] G = -\partial_\sigma G \tag{14}$$

where the coefficient of the first-order derivative is given by

$$g(r(r')) = \frac{-2r+3M}{r^2\sqrt{1-\frac{2M}{r}}} \tag{15}$$

depending on r' only implicitly. Now substitute for the heat kernel the following expression

$$G = \tilde{G}Y(\Omega)Y^*(\Omega')e^{\frac{1}{2}\int g dr'} \tag{16}$$

where $Y(\Omega)$ is a spherical harmonic, in order to obtain

$$\left[\partial_{t'}^2 - \partial_{r'}^2 - \frac{1}{2}\partial_{r'}g + \frac{1}{4}g^2 - \frac{l(l+1)}{r^2} + m^2 + \xi R\right]\tilde{G} = -\partial_\sigma \tilde{G} \tag{17}$$

and continue by substituting

$$\tilde{G}(t', r', \tilde{t}', \tilde{r}'; \sigma) = G_0^{2d}(t', r', \tilde{t}', \tilde{r}'; \sigma)e^T \equiv G_0^{2d}(t', r', \tilde{t}', \tilde{r}'; \sigma)e^{\sum_{n=0}^{\infty}\tau_n(t', r')\sigma^n} \tag{18}$$

where $G_0^{2d}(t', r')$ is the flat heat kernel in two dimensions, solving the equation

$$\left(\partial_{t'}^2 - \partial_{r'}^2\right)G_0^{2d}(t', r'; \sigma) = -\partial_\sigma G_0^{2d}(t', r', \sigma) \tag{19}$$

subject to the boundary condition $G_0^{2d}(x, \tilde{x}, 0) = \delta(\Delta(x, \tilde{x}))$. This gives the following equation for T;

$$\partial_{t'}^2 T + (\partial_{t'}T)^2 - \partial_{r'}^2 T - (\partial_{r'}T)^2 - \frac{1}{2}\partial_{r'}g + \frac{1}{4}g^2 - \frac{l(l+1)}{r^2} + m^2 + \xi R = -\partial_\sigma T \tag{20}$$

The boundary condition determines the zeroth Taylor coefficient of T as

$$\tau_0 = -\frac{1}{2}\int g dr' \tag{21}$$

the equation for T yields the next coefficient as

$$\tau_1 = -\left[-\partial_{r'}^2\tau_0 - (\partial_{r'}\tau_0)^2 - \frac{1}{2}\partial_{r'}g + \frac{1}{4}g^2 - \frac{l(l+1)}{r^2} + m^2 + \xi R\right]$$
$$= \frac{l(l+1)}{r^2} - m^2 + \xi R \approx \frac{l(l+1)}{r^2} \tag{22}$$

where m^2 (m is the mass of the particle as measured in Planck units) can be neglected, except perhaps for $r \to \infty$ where it will cut of the integral in which it occurs, cf. Eqs. (27) and (28). Inserting the Taylor series for T into equation gives a recursion formula for subsequent coefficients;

$$\tau_{n+1} = \frac{1}{n+1}\left[\partial_{r'}^2\tau_n + \sum_{n'=o}^{n}\partial_{r'}\tau_{n'}\partial_{r'}\tau_{n-n'}\right] \quad n \geq 2 \tag{23}$$

thus giving the following coefficient ($\partial_{r'} = \sqrt{1 - \frac{2m}{r}}\partial_r$);

$$\tau_2 = \frac{1}{2}\left[\partial_{r'}^2\tau_1 - g\partial_{r'}\tau_1\right]$$
$$= \frac{1}{2}\left(1 - \frac{2M}{r}\right)\partial_r^2\tau_1 + \frac{r-M}{r^2}\partial_r\tau_1$$
$$= \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left[\frac{6l(l+1)}{r^4} - \xi\partial_r^2 R\right] + \frac{r-M}{r^2}\left[-\frac{2l(l+1)}{r^3} - \xi\partial_r R\right]$$
$$= r^{-4}\left(1 - 4\frac{M}{r}\right)l(l+1) \tag{24}$$

and the next one

$$\tau_3 = \frac{1}{3}\left[\left(1 - \frac{2M}{r}\right)\partial_r^2\tau_2 - g\sqrt{1 - \frac{2M}{r}}\partial_r\tau_2 + \left(1 - \frac{2M}{r}\right)(\partial_r\tau_1)^2\right)$$
$$\approx \frac{4}{3}r^{-6}\left(1 - \frac{2M}{r}\right)l^2(l+1)^2 \tag{25}$$

and so forth. This is the only coefficients that we shall need explicitly. Also, the part of τ_3 that belongs to 2 loop order contributions has been omitted in the approximation of Eq. (16). That the omitted part is of higher order is most easily seen by restoring the ξR term in τ_1 before calculating τ_2, τ_3 and noting that one then gets terms of the form ∂_r^{2n}/R^{n-1} with n larger and larger. By omitting these higher-order contributions one thus implicitly assumes the background to be slowly varying (as compared to its strength). Also, one should probably not venture beyond 1 loop order within the given framework, cf. Eq. (27) later, because continued differentiations of the r^{-2} term of τ_1 will produce ever larger coefficients.

The effective action of the real scalar field is given by Eqs. (4) and (5);

$$\Gamma_{\text{scalar}} = \frac{1}{2}\sum_{lm}\int \sqrt{g}d^4x \left.\frac{d}{ds}\right|_{s=0}\int_0^\infty d\sigma \frac{\sigma^{s-1}}{\Gamma(s)}G \tag{26}$$

We are interested in the Casimir energy density and thus the effective Lagrangian, so averaging over angles gives

$$\begin{aligned}
\mathcal{L}_{\text{eff}}(r) &= \frac{1}{4\pi r^2}\sum_{lm}\frac{1}{2}\int d\Omega r^2 \left.\frac{d}{ds}\right|_{s=0}\int_0^\infty \frac{\sigma^{s-1}}{\Gamma(s)}G(x,\tilde{x};\sigma)\\
&= \sum_{lm}\frac{1}{8\pi}\left.\frac{d}{ds}\right|_{s=0}\int_0^\infty d\sigma\frac{\sigma^{s-1}}{\Gamma(s)}\int d\Omega Y(\Omega)Y^*(\Omega')e^{\frac{1}{2}\int g dr'}G_0^{(2d)}(t',r')e^T\\
&= \sum_{l}\frac{1}{8\pi}\left.\frac{d}{ds}\right|_{s=0}\int_0^\infty d\sigma\frac{\sigma^{s-1}}{\Gamma(s)}\frac{2l+1}{4\pi}\frac{1}{4\pi\sigma}e^{-\frac{\Delta^2(x,x)}{4\sigma}}\exp\left(\sum_{n=1}\tau_n\sigma^n\right)\\
&\approx \sum_{l}\frac{2l+1}{2(4\pi)^3}\left.\frac{d}{ds}\right|_{s=0}\int_0^\infty d\sigma\frac{\sigma^{s-2}}{\Gamma(s)}e^{\tau_1\sigma}(1+\tau_2\sigma^2+\tau_3\sigma^3\\
&\quad +\cdots+\tau_n\sigma^n+\cdots)\\
&= \sum_{l}\frac{2l+1}{2(4\pi)^3}\left[\tau_1-\tau_1\ln(-\tau_1)-\frac{\tau_2}{\tau_1}+\frac{\tau_3}{(-\tau_1)^2}\right.\\
&\quad \left.+\cdots\sum_{n=4}^{\infty}\frac{\tau_n}{(-\tau_1)^{n-1}}(n-2)!\right]\\
&\approx \sum_{l}\frac{2l+1}{2(4\pi)^3}\left[\tau_1-\tau_1\ln(-\tau_1)-\frac{\tau_2}{\tau_1}+\frac{\tau_3}{(-\tau_1)^3}\right]
\end{aligned}\tag{27}$$

where only the lowest-order correction to the contribution of the smooth classical background has been kept (i.e. tree level + 1 loop order, essentially). Note the occurrence of $(n-2)!$ in the continued expansion which make further expansion uninteresting (and probably un-sound) because the coefficients are so complicated that regularising the sum seems impossible. However, the framework of quantum field theory in curved space should be sound enough to one loop order, and we will have to be content with that:

Explicitly, for the Schwarzschild metric, collecting terms with similar dependence on l and noting that $\xi R = 0$ everywhere except at the origin, this becomes

$$\mathcal{L}_{\text{eff}}(r) = \sum_{l=0}^{\infty}\left[\left(\frac{2(1-\frac{2M}{r})}{3(4\pi)^3r^2}\right)l^0\left(-\frac{\ln(r)}{(4\pi)^3r^2}-\frac{1}{2(4\pi)^3r^2}+\frac{4(1-\frac{2M}{r})}{3(4\pi)^3r^2}\right)l\right.$$

$$
\begin{aligned}
&+\left(-\frac{3\ln(r)}{(4\pi)^3r^2}-\frac{3}{2(4\pi)^3r^2}\right)l^2\\
&+\left(-\frac{2\ln(r)}{(4\pi)^3r^2}-\frac{1}{(4\pi)^3r^2}\right)l^3\\
&+\left(\frac{3(1-\frac{2M}{r})}{2(4\pi)^3r^2}-\frac{r-M}{(4\pi)^3r^3}\right)\frac{l}{l^2+l}\\
&+\left(\frac{9(1-\frac{2M}{r})}{2(4\pi)^3r^2}-\frac{3(r-M)}{(4\pi)^3r^3}\right)\frac{l^2}{l^2+l}\\
&+\left(\frac{3(1-\frac{2M}{r})}{(4\pi)^3r^2}-\frac{2(r-M)}{(4\pi)^3r^3}\right)\frac{l^3}{l^2+l}\\
&+\left(\frac{1}{2(4\pi)^3r^2}\right)l\ln(-l^2-l)\\
&+\left(\frac{3}{2(4\pi)^3r^2}\right)l^2\ln(-l^2-l)\\
&+\left(\frac{1}{(4\pi)^3r^2}\right)l^3\ln(-l^2-l)\Bigg]
\end{aligned}
\tag{28}
$$

The first three terms easily renormalise as

$$
\left(\sum_l l^{-s}\right)_{\text{reg}}=\zeta(s) \tag{29}
$$

where $\zeta(s)$ is the Riemanian zeta-function with the relevant values being $\zeta(0)=-\frac{1}{2}$, $\zeta(-1)=-1/12$, $\zeta(-2)=-0$ and $\zeta(-3)=1/120$ (Gradstheyn and Ryzhik, 1980). The rest of the terms are renormalised using the modified Abel-Plana formula (Grib *et al.*, 1994) (taking out the $l=0$ term before use),

$$
\text{reg}\sum_{n=1}^{\infty}F(n)=-\frac{1}{2}F(0)+i\int_0^{\infty}\frac{F(it)-F(-it)}{e^{(2\pi t)-1}}dt \tag{30}
$$

and, where necessary, replacing infinite values by principal ones (Blau *et al.*, 1988) (and using $\ln(-l^2-l)=\ln(-1)+\ln(l)+\ln(l+1)$ where $\ln(-1)=i\pi$). The renormalised free energy thus becomes

$$
\begin{aligned}
\mathcal{L}_{\text{eff}}(r)=&\left[-\frac{1}{2}\left(\frac{2\left(1-\frac{2M}{r}\right)}{3(4\pi)^3r^2}\right)-\frac{1}{12}\left(-\frac{\ln(r)}{(4\pi)^3r^2}-\frac{1}{2(4\pi)^3r^2}+\frac{4\left(1-\frac{2M}{r}\right)}{3(4\pi)^3r^2}\right)\right]\\
&+\frac{1}{120}\left(-\frac{2\ln(r)}{(4\pi)^3r^2}-\frac{1}{(4\pi)^3r^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ (0.5 - i0,0772)\left(\frac{3\left(1-\frac{2M}{r}\right)}{2(4\pi)^3r^2} - \frac{r-M}{(4\pi)^3r^3}\right)\\
&+ 0.0161\left(\frac{9\left(1-\frac{2M}{r}\right)}{2(4\pi)^3r^2} - \frac{3(r-M)}{(4\pi)^3r^3}\right)\\
&+ i0,000612\left(\frac{3\left(1-\frac{2M}{r}\right)}{(4\pi)^3r^2} - \frac{2(r-M)}{(4\pi)^3r^3}\right)\\
&+ (-0.0881 - i0.262)\left(\frac{1}{2(4\pi)^3r^2}\right)\\
&+ (-0.0230)\left(\frac{3}{2(4\pi)^3r^2}\right)\\
&+ (-0.00538 + i0.0305)\left(\frac{1}{(4\pi)^3r^2}\right)\Bigg]\\
= &\frac{1}{15(4\pi)^3}\frac{\ln(r)}{r^2} + \frac{-0.224 + i0,211}{(4\pi)^3}\frac{1}{r^2} + \frac{0.207 + i0.179}{(4\pi)^3}\frac{M}{r^3}\\
\equiv &\,\alpha\frac{\ln(r)}{r^2} + \beta\frac{1}{r^2} + \eta\frac{M}{r^3} + \cdots + \text{constant}\frac{M^n}{r^{n+2}} + \cdots
\end{aligned}
\tag{31}
$$

to (a little above) one loop order. Proceed to determine the energy–momentum tensor by

$$
T_{\mu\nu} = \frac{\delta\Gamma}{\delta g^{\mu\nu}} = \frac{\delta\Gamma}{\delta M}\frac{\delta M}{\delta g^{\mu\nu}} = \frac{\partial\Gamma}{\partial M}\frac{\partial M}{\partial g^{\mu\nu}} = \left(\eta\frac{1}{r^3} + 2\gamma\frac{M}{r^4}\right)\left(\frac{\partial g^{\mu\nu}}{\partial M}\right)^{-1} \tag{32}
$$

Here

$$
\frac{\partial g^{\mu\nu}}{\partial M} = \begin{pmatrix} \frac{2}{r\left(1-\frac{2M}{r}\right)^2} & 0 & 0 & 0\\ 0 & \frac{2}{r} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}
$$

is not invertible but due to the fact that space does not curve in angular directions we invert the ‘invertible block’ (i.e. symmetry forbids a pressure in angular directions) and put equal to zero the (2,2) and (3,3) components in the inverted matrix thus obtaining for the following Casimir contribution to the energy–momentum tensor

$$T_{\mu\nu} = \left(\eta \frac{1}{r^2} + \cdots + n \cdot \text{constant} \frac{M^{n-1}}{r^{n+1}}\right) \begin{pmatrix} \frac{1}{2}\left(1 - \frac{2M}{r}\right)^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

the T_{00} component is shown in Fig. 1. Because this result forms the basis for the analogous calculation for fermions notes pertaining to the earlier result will be postponed till the end of next section.

3. THE ZERO-POINT ENERGY OF A SPIN 1/2 FERMION FIELD FOR THE SCHWARZSCHILD METRIC

The zeta-function of an operator A is related to the zeta-function of its square by

$$\zeta_{A^2}(s) = \sum_{\lambda} (\lambda^2)^{-s} = \zeta_A(2s) \tag{35}$$

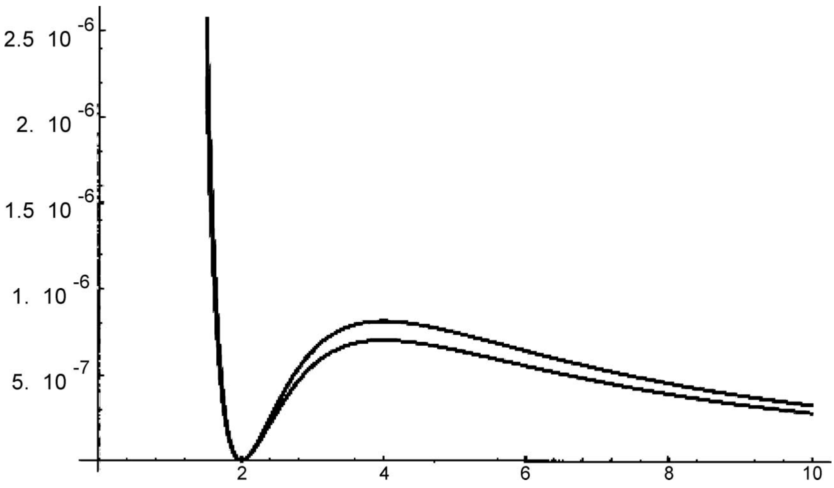


Fig. 1. Real (lower curve) and imaginary (upper curve) parts of T_{00}^{scalar} from just within the Schwarzschild radius ($r = 1.5M$) to $r = 10M$. Within the Schwarzschild radius T_{00}^{scalar} diverges as r^{-2}.

where λ is the eigenvalues of the operator. The Dirac operator of a fermion field coupled to a gauge field is

$$D_m = e_m^\mu \left(\partial_\mu + \frac{i}{2}\omega_\mu^{pq}(x) X_{pq} + i g A_\mu^a(x) T_a \right) \tag{36}$$

(which is both gauge and Lorentz covariant). Here e_m^μ is the vierbein (the local coordinate frame of a freely falling observer), $\omega_\mu^{pq}(x)$ is the spin connection being the gravitational analogue of the gauge field $A_\mu^a(x)$ and X_{pq} the corresponding Lorentz group ($SO(3,1)$) generators analogous to the gauge group generators T_a. Greek indices refer to curvilinear coordinates while small latin indices from the last half of the alphabet refer to local Lorentz coordinates. Small latin letters from the beginning of the alphabet will be used to denote gauge indices.

Representing the SO (3,1) generators in terms of the sigma matrices, $X_{pq} \equiv \sigma_{pq} = \frac{i}{4}[\gamma_p, \gamma_q]$, one obtains, for the derivative squared;

$$D^2 = \left(\Box + \xi_f R + m^2 + g\eta^{pq} A_p^a A_q^b T_a T_b\right) \cdot \mathbf{1}_4 + 2g\sigma^{pq} F_{pq}^a T_a + \mathcal{G}(A) \tag{37}$$

where $\mathbf{1}_4$ is the four-dimensional unit matrix and where

$$\mathcal{G}(A) = 0 \tag{38}$$

is an allowed gauge (Bormann and Antonsen, 1995).

It is possible, in principle, to include the gauge coupling using the meanfield approach introduced briefly in the next section (i.e. replacing the gauge field with its mean value in the Dirac operator), using either the earlier relationship (Eqs. (37) and (38)) or the relation of the fermion heat kernel to the scalar one. Here, for simplicity, we will only consider the case $\langle A_m^a \rangle = 0$, i.e. non-interacting fermions ($\langle A_m^a \rangle \neq 0$ is a higher-order effect). One can then proceed by doing the same calculations as in the scalar case, leading to

$$\begin{aligned} \mathcal{L}_{\text{eff}}(r) &= -\frac{1}{2}\frac{1}{4\pi r^2}\frac{d}{ds}|_{s=0}\zeta\nabla(s) \\ &\approx -\text{Tr}\sum_l \frac{2l+1}{2(4\pi)^3}\frac{d}{ds}\bigg|_{s=0}\int_0^\infty \frac{\sigma^{s-2}}{\Gamma(s)}e^{\tau_1\sigma}(1+\tau_2\sigma^2+\tau_3\sigma^3)\mathbf{1}_4 \\ &= -\sum_l \frac{2l+1}{2(4\pi)^3}\text{Tr}\left[\tau_1 - \tau_1\ln(\tau_1) - \frac{\tau_2}{\tau_1} + \frac{\tau_3}{\tau_1}\right] \\ &= -4\mathcal{L}_{\text{eff}}^{\text{scalar}}\Box + \xi_{f^R}(r) \end{aligned} \tag{39}$$

where in obtaining the last line we have taken note of Eq. (36), giving a factor 1/2, the trace over spinor indices gives a factor 4 and the fact that we are considering a complex Grassman field gives a factor -2, cf. Eq. (4). As $\xi_f R = 0$ this quantity has already been calculated in the previous section and the energy–momentum

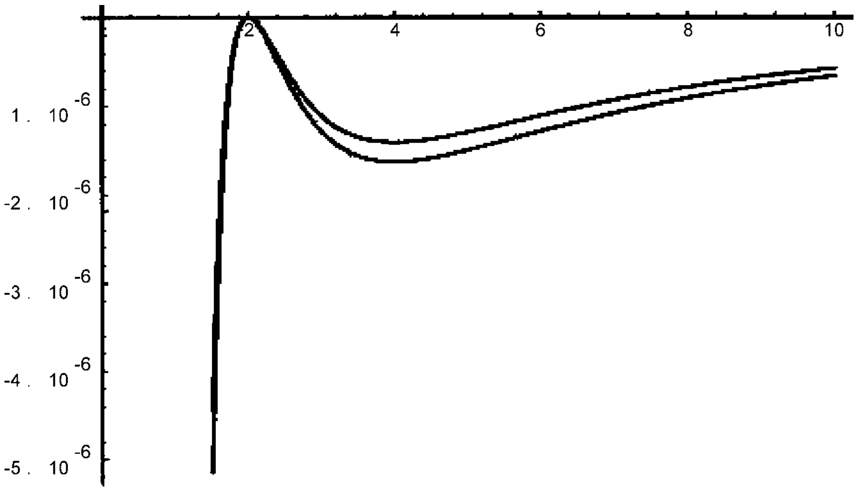


Fig. 2. Real (upper curve) and imaginary (lower curve) parts of T_{00}^{fermion} from just within the Schwarzschild radius ($r = 1.5M$) to $r = 10M$. Within the Schwarzschild radius T_{00}^{fermion} diverges as $-r^{-2}$.

tensor likewise becomes

$$T_{\mu\nu}^{\text{Dirac}} = -2\left(\eta\frac{1}{r^2} + \cdots + n \cdot \text{constant}\frac{M^n}{r^{n+2}}\right)\begin{pmatrix} \frac{1}{2}\left(1 - \frac{2M}{r}\right)^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

with $\eta = (-0.224 + i0.211)/(4\pi)^3$. A plot of T_{00}^{Dirac} is given in Fig. 2.

As already mentioned, the earlier expansion of the contribution of the zero-point fluctuations to the energy–momentum tensor is readily interpretated as equivalent to a loop-expansion, and one now sees that at the tree level one gets no contribution (its contribution disappears when varying the effective Lagrangian with respect to the metric). This is no surprise, partly because this is the classical limit, partly because a vertex is a local quantity and so is not affected by propagating virtual particles.

The 1 loop level contributes a term ($\propto (M^0/r^2)(l - \frac{2M}{r})^2$) that is non-zero as M goes to zero indicating a topological origin. 2 loop and higher order contributes mass-dependent terms that probably cannot be determined reliably by the earlier method. But as far as these terms can be taken as an indication of the higher-order contributions, note that these become more and more peaked at the origin, the higher the loop order.

When one does not take into account the contribution of the zero-point fluctuations to the energy–momentum tensor it is identically zero except at the origin (the energy–momentum tensor is $T_{\mu\nu} = M\delta(\mathbf{r})$, even if the Newtonian potential is $\propto M/r^2$). This no longer is the case when quantum fluctuations are included: For large distances, the Casimir contribution behaves as r^{-2} thus, if un-screened, adding up to an infinite contribution (ignoring higher-order terms). Then of course, back-reaction on the metric was not included – some kind of equilibrium might exist.

Also the Casimir energy contribution is complex. One may be tempted to consider only the real value of this contribution, but as we shall see when treating gauge bosons, the question of whether the energy–momentum tensor is complex may be linked to the topology (inside vs. outside the Schwarzschild radius) as well as to the size of gauge coupling constants and Casimir mean fields, making such an approach dubious.

One is better off considering energy complex and then interpreting the imaginary part of the energy as signaling particle creation.[5] One more thing to note about the earlier explanation, namely that within the present framework one unfortunately cannot have confidence in terms of higher order than indicated earlier. This is annoying as terms of the type M^n/r^{n+2} are potentially interesting for large masses and small distances giving what might be a major contribution to the energy–momentum tensor (including a large radial pressure that might be positive or negative depending on the coefficients in the earlier expansion), a contribution that should have been incorporated when one determined the metric in the first place (a sort of back-reaction, perhaps having the mass contribution bootstrap). The pressure contribution might be interesting if one considers the Schwarzschild metric an approximation to a general mass because in the early universe mass is in a sense within its own Schwarzschild radius. (It is entirely possible that $T_{\mu\nu}^{\text{Casimir}} \approx T_{\mu\nu}$ for $r \to 0$, in a full theory.)

As noted earlier, an interesting fact is that if one lets the mass generating the space–time go to zero one still has a (complex) contribution to the energy–momentum tensor which we will have to ascribe to the difference in topology to the Minkowski space. The size of this contribution is given in Planck units (multiply by $F_{\text{Planck}} \sim 10^{56}N$ to get the size in SI units), the size indicating that topology changes do not come easy. They should occur, though, at scales somewhat below the Planck one.

When discussing implications of the Casimir contribution to the energy–momentum tensor, we should include the full standard model. However, to calculate the spin 1 contribution we need the Lorentz condition which is at odds with the gauge condition (39). This problem probably could be migrated to second loop level. But this would enhance the complexity of the spin 1 calculation which is

[5] To be precise, the number of particles created is $\propto \exp(-2\text{Im}(\mathcal{L}_{\text{eff}}))$ (Grib *et al.*, 1994).

notably more complicated than the preceding cases, so proceed by considering just a pure Yang–Mills theory.

4. THE ZERO-POINT ENERGY OF A SPIN 1 GAUGE BOSON FIELD FOR THE SCHWARZSCHILD METRIC

For gauge theory containing fermions coupled to a non-Abelian gauge field one has the following generating functional

$$Z = \int DA_\mu \int D\psi D\bar{\psi} e^{-\frac{1}{4}\int F^a_{mn} F^{mn}_a dx^\mu + i\int \bar{\psi}\gamma^m D_m \psi dx^\mu} \tag{41}$$

where the field strength tensor is given by (Ramond, 1989)

$$F^a_{mn} = e^\mu_m e^\nu_n \left(\partial_\mu A^a_\nu + \partial_\nu A^a_\mu i g f^a_{bc} A^b_\mu A^c_\nu\right) \tag{42}$$

and the gauge invariant and Lorentz covariant derivative is given by Eq. (36).

As the fermion part of the action contains reference to the gauge field one *a priori* cannot carry out the two integrations independently. To remedy this, make a mean field approximation to A_μ in the fermionic integral, as well as in the higher-order terms of the bosonic integral (see later) and proceed by considering the bosonic and the fermionic parts independently. The fermionic case was discussed in the previous section so consider the bosonic part of the generating functional,

$$Z = \int DA_\mu e^{-\frac{1}{4}\int F^a_{mn} F^{mn}_a dx^\mu} \tag{43}$$

which can (using commutation relations, suitable normalisation and the Lorentz condition) be given the form

$$Z = \int DA_\mu e^{-\int d^4x A^b_m \frac{g^2}{4}[-\delta^a_b \delta^m_n \partial_p \partial^p + \delta^a_b (\partial_n e^{m\mu} - \partial^m e^\mu_n) e^p_\mu \partial_p + g f_b{}^a{}_c (\partial_n A^{mc} - \partial^m A^c_n) + \frac{1}{2}\delta^m_n g^2 f_{ebc} f^a_{dc} A^e_p A^{pd}] A^n_a} \tag{44}$$

and, in order to perform this path integral, make it Gaussian by choosing the following mean field approximation:

$$\begin{aligned} Z &= \int DA_\mu e^{-\int d^4x A^b_m \frac{g^2}{4}[-\delta^a_b \delta^m_n \partial_p \partial^p + \delta^a_b (\partial_n e^{m\mu} - \partial^m e^\mu_n) e^p_\mu \partial_p + \langle g f_b{}^a{}_c (\partial_{nm} A^{mc} - \partial^m A^c_n)\rangle + \langle \frac{1}{2}\delta^m_n g^2 f_{ebc} f_d{}^a{}_c A^e_p A^{pd}\rangle] A^n_a} \\ &\equiv \int DA_\mu e^{-\int d^4x A^b_m M^{am}_{bn} A^n_a} \\ &= (\det\ M^{am}_{bn})^{-\frac{1}{2}} \end{aligned} \tag{45}$$

To determine this path integral, once again use the heat kernel method, this time for the differential operator of the earlier path integral. Thus, consider the equation

$$\frac{g^2}{4}\Bigg[\delta^a_b\delta^m_r\partial_p\partial^p - \partial^a_b\big(\partial_r e^{m\mu} - \partial^m e^\mu_r\big)e^p_\mu\partial_p - \big\langle g f^a_{b\ c}\big(\partial_r A^{mc} - \partial^m A^c_r\big)\big\rangle - \left\langle\frac{1}{2}\delta^m_r g^2 f_{ebc} f^{\ a}_{d\ c} A^e_p A^{pd}\right\rangle\Bigg] G^{r(a)}_{n(b)}(x,\tilde{x},\sigma) = -\partial_\sigma G^{m(a)}_{n(b)}(x,\tilde{x};\sigma) \tag{46}$$

or, in short hand notation

$$\frac{g^2}{4}\big[\delta^a_b\delta^m_k\partial_p\partial^p - \delta^a_b\big(\partial_k e^{m\mu} - \partial^m e^\mu_k\big)e^p_\mu\partial_p - f^{m(a)}_{k(b)}(\langle A\rangle)\big]G^{k(a)}_{n(b)}(x,\tilde{x};\sigma) = -\partial_\sigma G^{m(a)}_{n(b)}(x,\tilde{x};\sigma) \tag{47}$$

The first-order term is eliminated by the substitution

$$G = \tilde{G}e^{\frac{1}{2}\int\left(\partial_n e^{m\mu} - \partial^m e^\mu_n\right)e^p_\mu dx_p} \tag{48}$$

whereupon $\tilde{G}$ is written as

$$\tilde{G}^{m(a)}_{n(b)}(x,\tilde{x};\sigma) = G^o\left(x,\tilde{x};\frac{g^2}{4}\sigma\right)\left(e^{T(x,\tilde{x}\sigma)}\right)^{(a)m}_{n(b)} \tag{49}$$

where G^o denotes the heat-kernel of $\Box_0 = \partial_p\partial^p$ and T is some matrix $(T)^{m(a)}_{n(b} = T^{m(a)}_{n(b)}$ which we expand as

$$T^{m(a)}_{n(b)}(x,\sigma) = \sum_{\nu=0}^{\infty}\tau^{(\nu)m(a)}_{n(b)}(x)\sigma^\nu \tag{50}$$

Due to the boundary condition, $G^{m(a)}_{n(b)}(x,\tilde{x};0) = \delta^a_b\delta^m_n\delta(x,\tilde{x})$, on the heat kernel;

$$G^{m(a)}_{n(b)}(x,\tilde{x};\sigma) = G^o(x,\tilde{x};\sigma)\big(e^{T(x,\tilde{x}\sigma)}\big)^{(a)m}_{n(b)}e^{\frac{1}{2}\int(\partial_n e^{m\mu} - \partial^m e^\mu_n)e^p_\mu dx_p} \tag{51}$$

the first coefficient of the expansion becomes

$$\partial^p\tau^{(0)m(a)}_{n(b)} = -\frac{1}{2}\delta^a_b\big(\partial_n e^{m\mu} - \partial^m e^\mu_n\big)e^p_\mu \tag{52}$$

Note in passing that the right hand side of this equation is proportional to the structure coefficent of the Lie algebra of the derivatives, ∂_μ, a measure of the space–time curvature (see, e.g. Ramond, 1989).

Utilising that along the diagonal $x = \tilde{x}$ the flat space heat kernel is constant, making its derivatives vanish, and Eqs. (47) and (51), the next coefficient becomes

$$\tau^{(1)m(a)}_{n(b)} = \left(\frac{g^2}{4}\right)^1\Bigg[\delta^b_a\partial_p\big[\big(\partial_n e^{m\mu} - \partial^m e^\mu_n\big)e^p_\mu\big] - \frac{1}{2}\delta^a_b\big[\big(\partial_k e^{m\mu} - \partial^m e^\mu_k\big)e^p_\mu\big]\big[\big(\partial_n e^{k\nu} - \partial^k e^\nu_n\big)e_{p\nu}\big] + f^{m(a)}_{n(b)}(\langle A\rangle)\Bigg] \tag{53}$$

Also one derives a recursion relation for the coefficients $\tau_{n(b)}^{(\nu)m(a)}$

$$\frac{g^2}{4}\left[\Box_0\tau_{n(a)}^{(\nu)m(a)} + \sum_{\nu'=0}^{\nu}\left(\partial_p\tau_{k(c)}^{(\nu-\nu')m(a)}\left(\partial^p\tau_{n(b)}^{(\nu')k(c)}\right)\right] = -(\nu+1)\tau_{n(b)}^{(\nu)m(a)}; \quad \nu \geq 2 \tag{54}$$

and subsequently the next coefficient is

$$\tau_{n(b)}^{(2)m(a)} \approx \left(\frac{g^2}{4}\right)^2\left[-\frac{1}{2}\Box_0 f_{n(b)}^{m(a)}(\langle A\rangle) - \frac{1}{2}\partial_p f_{k(b)}^{m(a)}(\langle A\rangle)\cdot\left(\partial_n e^{k\mu} - \partial^k e_n^{\mu}\right)e_\mu^p - \frac{1}{2}\left(\partial_k e^{m\mu} - \partial^m e_k^{\mu}\right)e_\mu^p\cdot\partial_p f_{n(b)}^{k(a)}(\langle A\rangle)\right] \tag{55}$$

The last coefficient that we list is

$$\tau_3 \approx \left(\frac{g^2}{4}\right)^3\left(\partial_p f_k^m\right)\left(\partial^p f_n^k\right) \tag{56}$$

For computational reasons we shall only work to this order and furthermore have kept only the terms (of τ_2 and τ_3) that seem most important when doing simulations, i.e. the most important pure curvature term, the most important pure gauge field terms and the most important gauge field-curvature coupling terms.[6]

Thus, use for the heat kernel the expression

$$G_{n(b)}^{m(a)}(x,x,\sigma) = G^o(x,x;\sigma)\left(e^{\tau_0+\tau_1\sigma^1+\tau_2\sigma^2+\tau_3\sigma^3}\right)_{n(b)}^{m(a)} \tag{57}$$

to find the effective action:

$$\Gamma = -\left(\ln\left(\det\left(M_{(b)n}^{(a)m}\right)\right)^{-1/2} = \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\zeta(s)$$

$$\approx \frac{1}{2}\sum_{m,l}\frac{d}{ds}\bigg|_{s=0}\int\sqrt{g}d^4x\int_0^\infty d\sigma \tag{58}$$

$$\frac{\sigma^{s-1}}{\Gamma(s)}\frac{1}{\left(4\pi\frac{g^2}{4}\sigma\right)^2}e^{\tau_1\sigma+\tau_2\sigma^2+\tau_3\sigma^3+\cdots+\tau_n\sigma^n+\cdots}Y_{lm}(\Omega)Y_{lm}^*(\Omega')$$

$$\approx \frac{1}{2}\sum_{m,l}\frac{d}{ds}\bigg|_{s=0}\int\sqrt{g}d^4x\int_0^\infty d\sigma \tag{59}$$

$$\mathrm{Tr}\frac{\sigma^{s-1}}{\Gamma(s)}\frac{1}{\left(4\pi\frac{g^2}{4}\sigma\right)^2}e^{\tau_1\sigma}(1+\tau_2\sigma^2+\tau_3\sigma^3+\cdots+\tau_n\sigma^n+\cdots)Y_{lm}(\Omega)Y_{lm}^*(\Omega')$$

[6] Note, however, that by discarding terms that are essentially higher and higher-order derivatives of the curvature one might lose a large contribution when close to singularities.

where the trace is both over Lorentz and gauge indices. From this one finds its density, the effective Lagrangian to be

$$\mathcal{L}_{\text{eff}}(r) \approx \text{Tr}\frac{1}{2}\left(\frac{g^2}{4}\right)^{-2}\sum_l \frac{d}{ds}\bigg|_{s=0}\frac{4^2}{4\pi r^2}\frac{2l+1}{4\pi}$$
$$\times \int_0^\infty d\sigma \frac{\sigma^{s-1}}{\Gamma(s)}\frac{1}{(4\pi\sigma)^2}e^{\tau_1\sigma}(1+\tau_2\sigma^2+\tau_3\sigma^3+\cdots+\tau_n\sigma^n+\cdots)$$
$$= \text{Tr}\left(\frac{g^2}{4}\right)^{-2}\frac{1}{(4\pi)^4}\sum_l(2l+1)\left[\frac{3}{4}\tau_1^2-\frac{1}{2}\tau_1^2\ln(-\tau_1)-\ln(-\tau_1)\tau_2-\frac{\tau_3}{\tau_1}\right.$$
$$\left. +\cdots+(n-3)!\frac{\tau_n}{\tau_1^{n-2}}+\cdots\right] \tag{60}$$

To perform the integration over the angular coordinates, we utilised the fact that the heat kernel formally solves a spherically symmetric differential equation and thus must be spherically symmetric itself. The only angular dependenies thus are those of the spherical harmonics which are taken care of using the relation $\Sigma_{m,l}\int d\Omega Y_{lm}(\Omega)Y^*_{lm}(\Omega)=\frac{2l+1}{4\pi}$. Note that, in the approximation earlier, each of the τ_n will have an angular dependency (see later). If, however, we were able to explicitly perform the sum (50) this angular dependency would cancel out. In this case performing the angular part of the integral as done earlier would be correct. For this reason we preferred this procedure to one of first inserting the τ_n, their angular dependencies included, and then performing the integration. To explicitly determine the coefficients, τ_1, τ_2, and τ_3, write the Schwarzschild metric as

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2 d\Omega \tag{61}$$

with

$$h(r) = 1 - \frac{2M}{\tau} \tag{62}$$

where M denotes the mass of the object generating the gravitational field. The vierbeins then read

$$e_0^a = \begin{pmatrix}\sqrt{h}\\0\\0\\0\end{pmatrix}\quad e_1^a = \begin{pmatrix}0\\h^{-1/2}\\0\\0\end{pmatrix}\quad e_2^a = \begin{pmatrix}0\\0\\r\\0\end{pmatrix}\quad e_3^a = \begin{pmatrix}h\\0\\0\\r\sin\theta\end{pmatrix} \tag{63}$$

Thus, the first coefficient explicitly becomes

$$\begin{aligned}
{\tau_1}_n^m &= \left(\frac{g^2}{4}\right)^1 \left(\eta^{mm'}\partial_\mu\left(e_n^\gamma\partial_\gamma e_{m'}^\mu - e_{m'}^\nu\partial_\nu e_n^\mu\right) + \eta^{mm'}\left(e_n^\gamma\partial_\gamma e_{m'}^\mu - e_{m'}^\nu\partial_\nu e_n^\mu\right)e_p^\delta\partial_\delta e_\mu^p\right.\\
&\quad \left. -\frac{1}{2}\eta^{kk'}\eta^{mm'}g_{\mu\nu}\left(e^\tau\partial_\tau e_{m'}^\mu - e_{m'}^\tau\partial_\tau e_k^\mu\right)\left(e_n^\delta\partial_\delta e_k^\nu - e_{k'}^\delta\partial_\delta e_k^\nu\right) + f_n^m(\langle A\rangle)\right)\\
&= \left(\frac{g^2}{4}\right)^1 \left(-\frac{1}{2}\begin{pmatrix} h^2(\partial_r h^{-1/2})^2 & 0 & 0 & 0\\ 0 & h^2(\partial_r h^{-1/2})^2 + 2\frac{h}{r^2} & \frac{h^{1/2}\cot\theta}{r^2} & 0\\ 0 & \frac{h^{1/2}\cot\theta}{r^2} & \frac{h+\cot\theta}{r^2} & 0\\ 0 & 0 & 0 & \frac{h+\cot\theta}{r^2}\end{pmatrix} + f_n^m(\langle A\rangle)\right)\\
&\approx \left(\frac{g^2}{4}\right)^1 \left(-\frac{1}{2}\begin{pmatrix} \frac{M^2}{r^2(r^2-2Mr)} & 0 & 0 & 0\\ 0 & \frac{M^2}{r^2(r^2-2Mr)} + 2\frac{1-\frac{2M}{r}}{r^2} & 0 & 0\\ 0 & 0 & \frac{1-\frac{2M}{r}}{r^2} & 0\\ 0 & 0 & 0 & \frac{1-\frac{2M}{r}}{r^2}\end{pmatrix} + f_n^m(\langle A\rangle)\right)
\end{aligned} \tag{64}$$

and the next one;

$$\begin{aligned}
{\tau_2}_m^n &= \left(\frac{g^2}{4}\right)^2 \left(-\frac{1}{2}\eta^{ab}e_a^\mu\left(\partial_\mu e_b^\nu\right)\partial_\nu f_n^m - \frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu f_n^m\right.\\
&\quad \left. -\frac{1}{2}\eta^{kk'}\left(\partial_\mu f_k^m\right)\left(e_n^\nu\partial_\nu e_{k'}^\mu - \eta_{k'}^\nu\partial_\nu e_n^\mu\right) - \frac{1}{2}\eta^{mm'}\left(e_k^\nu\partial_\nu e_{m'}^\mu - e_{m'}^\nu\partial_\nu e_k^\mu\right)\left(\partial_\mu f_n^k\right)\right)\\
&= \left(\frac{g^2}{4}\right)^2 \left(-\frac{1}{2\left(1-\frac{2m}{r}\right)}\partial_t^2 f_n^m + \frac{1}{2}\left(1-\frac{2m}{r}\right)\partial_r^2 f_n^m + \frac{m}{2r^2}\partial_r f_n^m\right.\\
&\quad + \frac{1}{2r^2}\partial_\theta^2 f_n^m + \frac{1}{2r^2\sin^2\theta}\partial_\phi^2 f_n^m\\
&\quad - \frac{M}{r^2\left(1-\frac{2M}{r}\right)}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\partial_t f_n^k - \frac{\sqrt{1-\frac{2M}{r}}}{2r^2}\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\partial_\theta f_n^k
\end{aligned}$$

$$
- \frac{1}{2r^2 \sin\theta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1-\frac{2M}{r}} \\ 0 & 0 & 0 & -\cot\theta \\ 0 & \sqrt{1-\frac{2M}{r}} & -\cot\theta & 0 \end{pmatrix} \partial_\phi f_n^k
$$

$$
- \frac{M}{r^2\left(1-\frac{2M}{r}\right)} \partial_t f_k^m \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{\sqrt{1-\frac{2M}{r}}}{2r^2} \partial_\theta f_k^m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
- \frac{1}{2r^2 \sin\theta} \partial_\phi f_k^m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1-\frac{2M}{r}} \\ 0 & 0 & 0 & -\cot\theta \\ 0 & \sqrt{1-\frac{2M}{r}} & \cot\theta & 0 \end{pmatrix} \Bigg) \Bigg)
$$

$$
\approx \left(\frac{g^2}{4}\right)^2 \left(\frac{1}{2}\left(1-\frac{2M}{r}\right)\partial_r^2 f_n^m + \frac{M}{2r^2}\partial_r f_n^m\right) \tag{65}
$$

and the last one that will be needed

$$
\begin{aligned}
\tau_3 &\approx \left(\frac{g^2}{4}\right)^3 \left(\partial_p f_k^m\right)\left(\partial^p f_n^k\right) = \left(\frac{g^2}{4}\right)^3 g^{\mu\nu}\left(\partial_\mu f_k^m\right)\left(\partial_\nu f_n^k\right) \\
&= \left(\frac{g^2}{4}\right)^3 \left(-\left(1-\frac{2M}{r}\right)\right)(\partial_r f)^{2m}_{n}
\end{aligned} \tag{66}
$$

where the final expression for the coefficients utilize that, because of the symmetry of the problem, the meanfield can only depend on the radial coordinate, thus $f_n^m = f_n^m(r)$. In a general gauge theory, the function $f_{(b)n}^{(a)m}$ is given by

$$
\begin{aligned}
f_{(b)n}^{(a)m} &= g f_{b\ c}^{\ a} \eta^{mm'}\left(e_n^\mu \partial_\mu \langle A_{m'}\rangle 1^c - e_{m'}^\mu \partial_\mu \rangle A_n^{(c)} \rangle 1^c\right) \\
&\quad + \delta_n^m g^2 f_{ebc} f_d^{ac} \delta^{ed} \langle A_p A_q \rangle \eta^{pq}
\end{aligned} \tag{67}
$$

where the fact that one does not *a priori* expect the meanfield to be colour asymmetric has been used (also this is verified later, to lowest order). As we shall only work to the lowest order, the first term will not contribute to the path integral (46) (otherwise we would break colour invariance, which on experimental grounds, one should not do lightly). The second term evaluates to, in the case of a *SU(N)* theory,

$$f^{(a)m}_{(b)n} \simeq \delta^m_n g^2 \langle A_p A_q \rangle \eta^{pq} C(N) \tag{68}$$

where $C(N)$ denotes the Casimir operator of the group.

The meanfield, used to determine f^m_n in in the earlier expression (through Eqs. (47) and (48)), is, by definition,[7]

$$\left\langle A^a_m(x) A^b_n(\tilde{x}) \right\rangle \equiv \frac{\int A^a_m(x) A^b_n(\tilde{x}) e^{iS} \mathcal{D}A}{\int e^{iS} \mathcal{D}A} \tag{69}$$

where S denotes the appropriate action. This meanfield has been determined in (Bormann and Antonsen, 1995). To lowest order the result is

$$\left\langle A^a_m(x) A^b_n(x) \right\rangle_{\text{reg}} = \delta^{ab} \tau_{1mn}(\gamma - 1) \tag{70}$$

This result can then in turn be made the starting point of an iteration using the full action (with non-zero mean field terms) to obtain a better approximation to the meanfield. However, I will just consider the lowest approximation, which from Eqs. (63) and (69) gives the following meanfield

$$\begin{aligned} A^{(a)}_m &= \frac{\sqrt{\gamma - 1}}{2r} \begin{pmatrix} \sqrt{-\dfrac{M^2}{r^2 - 2Mr}} \\ \sqrt{\dfrac{M^2}{r^2 - 2Mr} + 2\left(1 - \dfrac{2M}{r}\right)} \\ \sqrt{1 - \dfrac{2M}{r}} + \cot(\theta) \\ \sqrt{1 - \dfrac{2M}{r}} + \cot(\theta) \end{pmatrix} \\ &\approx \frac{\sqrt{\gamma - 1}}{2r} \begin{pmatrix} \sqrt{-\dfrac{M^2}{r^2 - 2Mr}} \\ \sqrt{\dfrac{M^2}{r^2 - 2Mr} + 2\left(1 - \dfrac{2M}{r}\right)} \\ \sqrt{1 - \dfrac{2M}{r}} \\ \sqrt{1 - \dfrac{2M}{r}} \end{pmatrix} \end{aligned} \tag{71}$$

[7] Where ever the mean value of a any odd power of the gauge fields occur, we probably should substitute $\sqrt{A^2}$ for A – keeping in mind though, that path integrating over A will kill any odd power of A (if the exponential of Eq. (46) was expanded in powers, which in turn was the reason for eliminating part of f^m_n earlier).

where the θ-dependence is an artefact due to the truncation of the series (50) and thus should be ignored in the calculations.[8]

Explicitly, from Eq. (68), one has

$$\begin{aligned} f^{(a)m}_{(b)n} &= \delta^m_n g^2 \delta^p_{q'} C(N)(\gamma - 1)\tau_1{}^{(a)q'}_{(b)p} \\ &= \delta^m_n g^2 C(N)(\gamma - 1)\mathrm{Tr}|_{\langle A\rangle=0}\tau_1 \\ &= -\frac{1}{2}\delta^m_n g^2 C(N)(\gamma - 1)\left[\frac{2M^2}{r^2(r^2-2Mr)} + \frac{4\left(1-\frac{2M}{r}\right)}{r^2}\right] \end{aligned} \tag{72}$$

and so τ_1 explicitly becomes

$$\begin{aligned} \tau_1 = -\frac{1}{2}\left(\frac{g^2}{4}\right) &\begin{pmatrix} \frac{M^2}{r^2(r^2-2Mr)} & 0 & 0 & 0 \\ 0 & \frac{M^2}{r^2(r^2-2Mr)} + 2\frac{1-\frac{2M}{r}}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1-\frac{2M}{r}}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1-\frac{2M}{r}}{r^2} \end{pmatrix} \\ &- \left(\frac{g^2}{4}\right)(\gamma-1)g^2C(N)\left[\frac{M^2}{(r^2-2Mr)r^2} + \frac{2\left(1-\frac{2M}{r}\right)}{r^2}\right]\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{73}$$

while τ_2 becomes

$$\begin{aligned} \tau_2 = \left(\frac{g^2}{4}\right)^2 &\left[-4\frac{M^2(2r-3M)(r-4M)}{r^6(r-2M)^2} + 4\frac{M^2}{r^5(r-2M)}\right. \\ &\left.- 12\,\frac{(r-4M)(r-2M)}{r^6} + \frac{M^3(2r-3M)}{r^6(r-2M)^2 + 2\frac{M(r-3M)}{r^6}}\right](\gamma-1)g^2C(N)\delta^m_n \end{aligned} \tag{74}$$

and τ_3 becomes

$$\tau_3 = -\left(\frac{g^2}{4}\right)^3\left[\left(1-\frac{2M}{r}\right)(\gamma-1)^2 g^4 C^2(N)\right]$$

[8] The series (50) is a factor in the (formal) solution to an spherically symmetric equation and thus the sum of the (infinitely many) contributions must be angle independent, even if individual terms, due to the angular dependancy of the vierbeins, has an angular dependancy.

$$\times \left[-2\frac{M^2(2r-3M)}{r^4(r-2M)^2} - 4\frac{r-3M}{r^4} \right]^2 \delta_n^m \tag{75}$$

The renormalised effective Lagrangian is found by performing the summation $\sum_{l=0}^{\infty}(2l+1) = 1 + \sum_{l=1}(2l+1) = 1 + 2\zeta(-1) + \zeta(0) = 1 + 2(-\frac{1}{2}) + (-\frac{1}{2}) = \frac{1}{3}$, and adding the Ghost contribution, $-2\mathcal{L}_{\text{eff}}^{\text{scalar,ren}}$, giving

$$\mathcal{L}_{\text{eff}}^{\text{ren}} = \text{Tr}\left(\frac{g^2}{4}\right)^{-2} \frac{1}{3(4\pi)^4} \left[\frac{3}{4}\tau_1^2 - \frac{1}{2}\tau_1^2 \ln(-\tau_1) - \ln(-\tau_1)\tau_2 - \tau_1^{-1}\tau_3\right] - 2\mathcal{L}_{\text{eff}}^{\text{scalar,ren}} \tag{76}$$

which is pretty straight forward to use because τ_1 is diagonal. However, the expression one gets is rather complicated and so only plots are presented. The plots were made using Matematica (Wolfram Research, 1997). The behaviour of the effective Lagrangian has been cut off towards the singularity where it diverges.

Figures 3a and 3b show large effects at the singularity and at the Schwarzschild radius. It is interesting that even there, where the gravitational field is very strong, the results show a strong dependency on the size of the gauge coupling constant. Also interesting is the fact that, as regards the imaginary part of the effective Lagrangian, for the gauge coupling constant being large, one gets a prominent contribution at longer distances. With what is probably a language abuse, one could speak of warping of gravitational and gauge field effects (for strong fields).

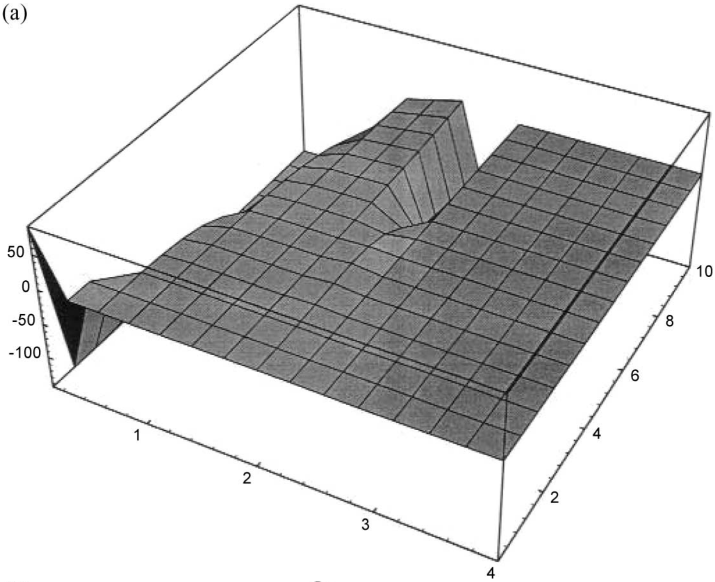
Because the effects are largest for large fields, drowning smaller ones, interesting effects do appear as one zooms in to have a higher-resolution view:

Focusing on the small g region, $g \in [0.01, 1.0]$, and also ignoring what goes on close to the Schwarzschild radius one sees some rather surprising local variations in the real part of the effective Lagrangian, a kind of resonances in g and r, a much more complicated behaviour than one would anticipate. These are evident both inside the Schwarzschild radius (Fig. 3c where $r/M \in [0.1, 1.9]$) and outside (Fig. 3e with $r/M \in [2.1, 6.0]$). Actually, there are a few very minor resonances in the imaginary part as well, as can be seen from the plots of $\exp(-2\text{Im}(\mathcal{L}_{\text{eff}}))$ (the particle creation rate) depicted in Figs. 4a and 4b. Again, the energy–momentum tensor is determined by

$$T_{\mu\nu} = \frac{\delta\Gamma_{\text{eff}}}{\delta M}\frac{\delta M}{\delta g^{\mu\nu}} \tag{77}$$

where $\frac{\delta g^{\mu\nu}}{\delta M}$ is given by Eq. (34). Plots of the energy–momentum tensor are given in Fig. 5, where again, the plots have been cut off towards the singularity where they diverge. The behaviour of course closely mimics that of the effective Lagrangian, except that the resonances stand out even further, dominating the high resolution plots of $\text{Im}(T_{00})$, Figs. 5c and 5e. Note, that the dependency of the effective Lagrangian on the coupling constant becomes more complicated as one goes to

(a)



(b)

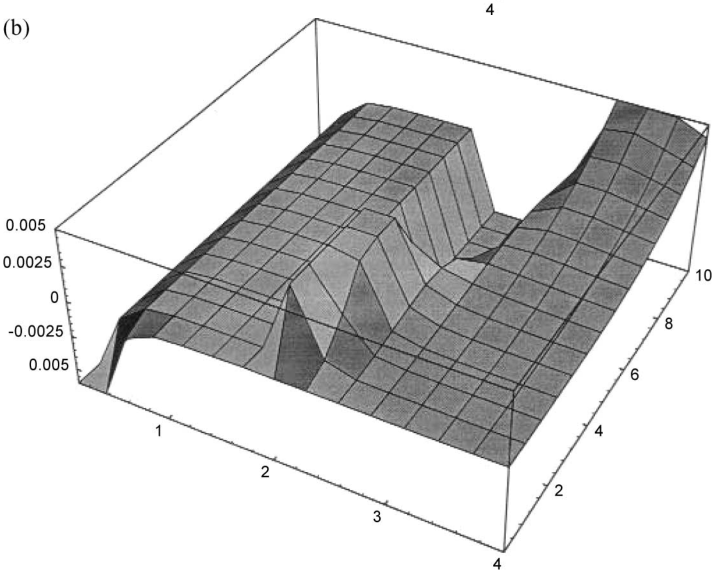
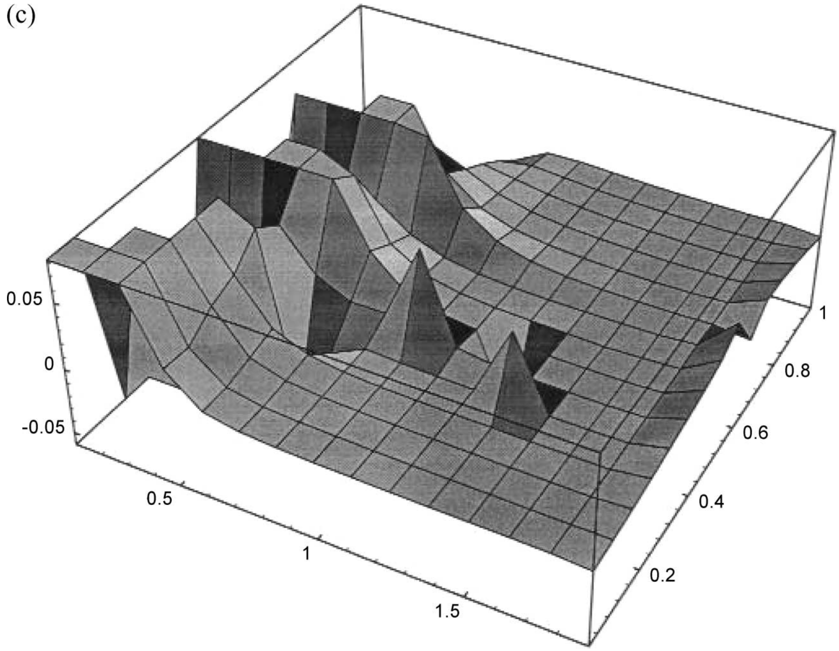


Fig. 3. The renormalised effective Lagrangian for a SU(3) field. Figures (a) and (b) show the real and imaginary parts, respectively, of $\mathcal{L}_{\text{eff}}^{\text{ren}}$ for $r \in$ [0.2M, 8M] and $g \in$ [0.01, 10]. Figures (c)–(f) zoom at the behaviour for small coupling constants, $g \in$ [0.01, 1.0]. Figures (c) and (d) show the real and imaginary parts of the effective Lagrangian within Schwarzschild radius, $r/M \in$ [0.1, 1.9]. Figures (e) and (f) show the real and imaginary parts of the effective Lagrangian outside the Schwarzschild radius, $r/M \in$ [2.1, 6.0].

(c)



(d)

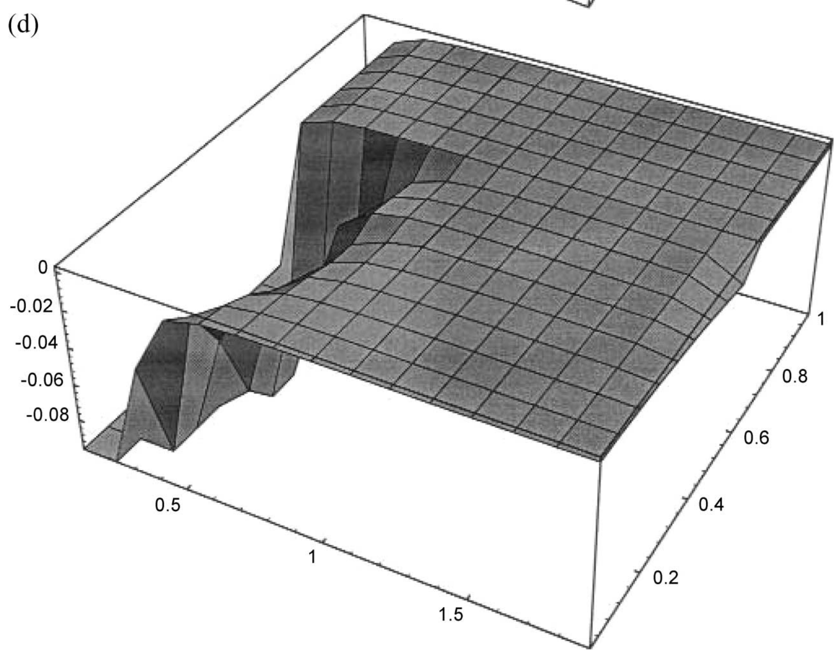
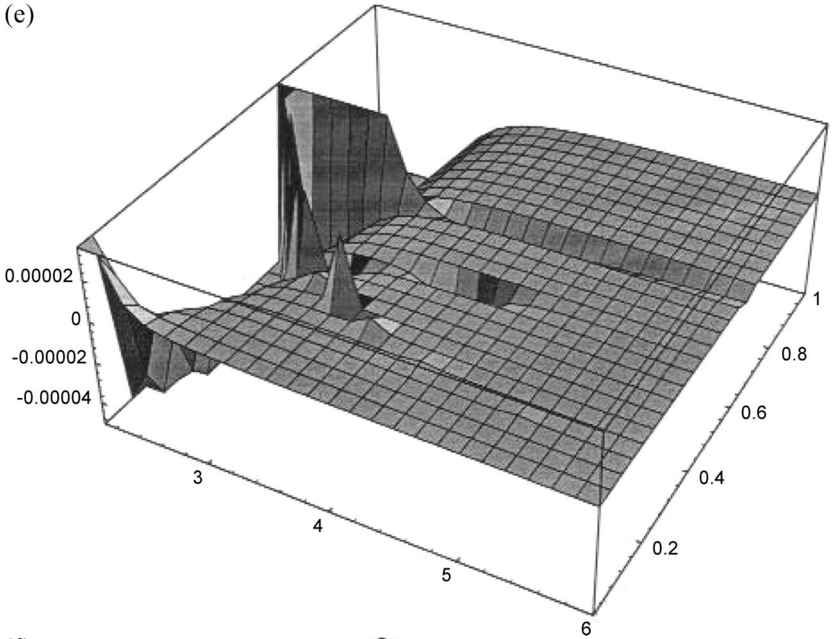


Fig. 3. Continued.

(e)



(f)

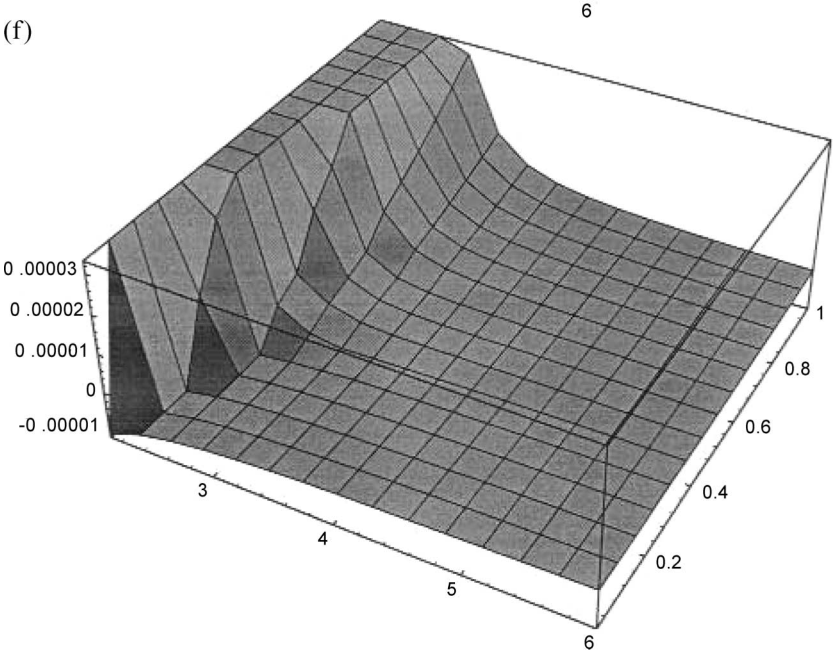
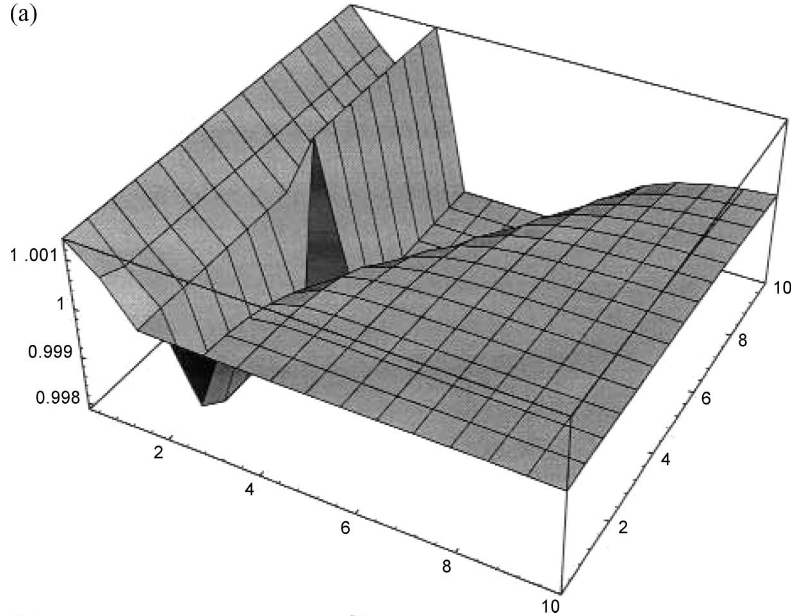


Fig. 3. Continued.

(a)



(b)

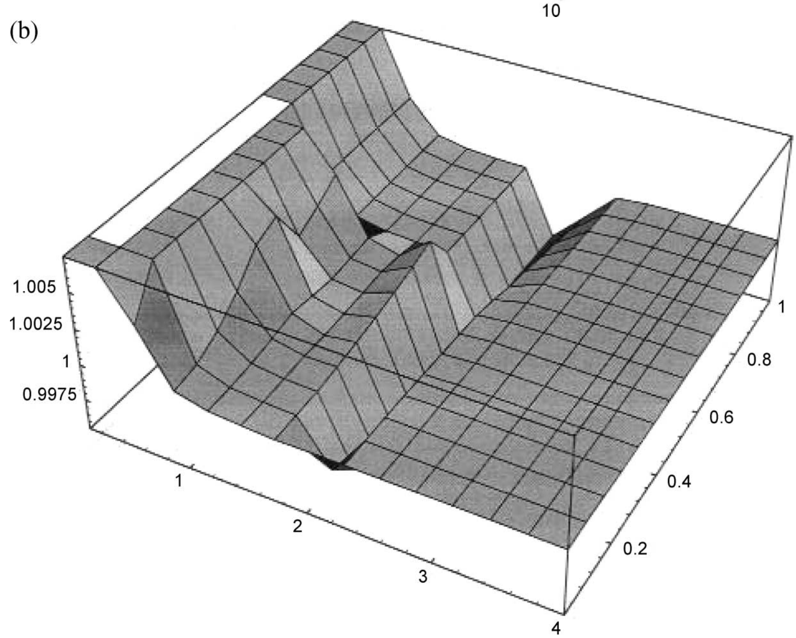


Fig. 4. The particle creation rate, exp(—2Im($\mathcal{L}_{\rm eff}$)), for a SU(3) field for (a) $r/M \in [0.2, 10.0]$ and $g \in [0.01, 10.0]$ and; (b) $r/M \in [0.2, 4.0]$ and $g \in [0.01, 1.0]$ (zooming on the small coupling region).

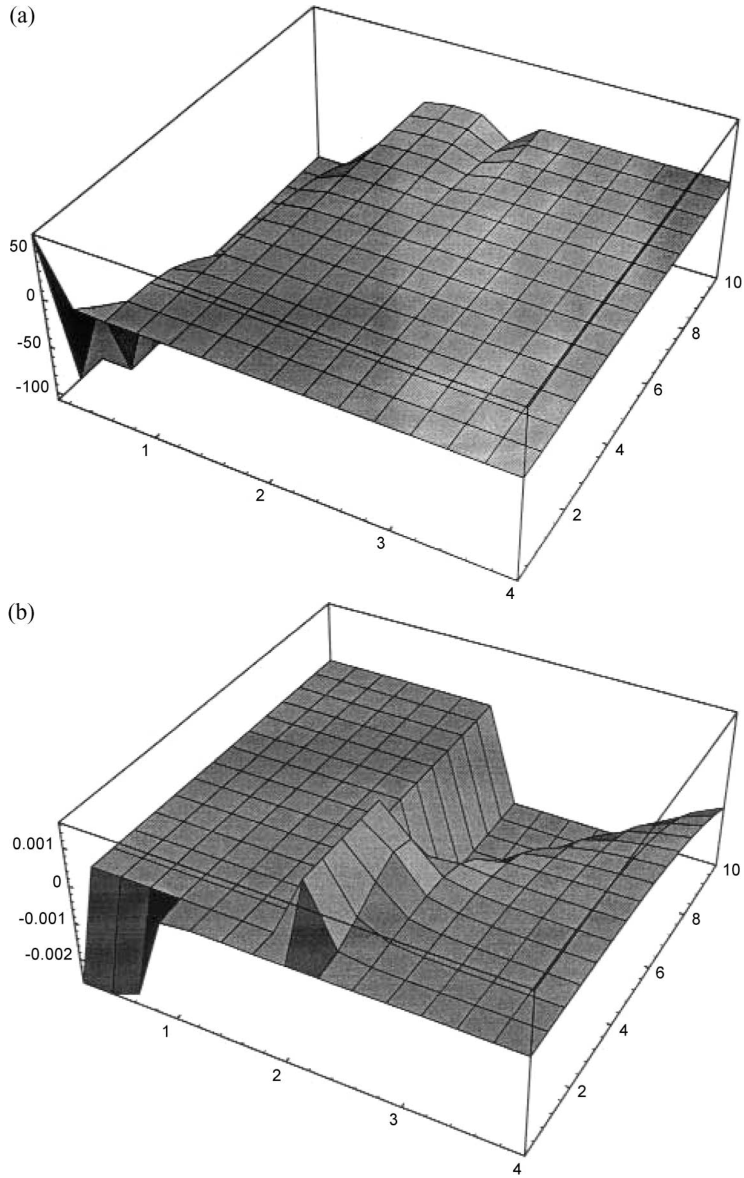
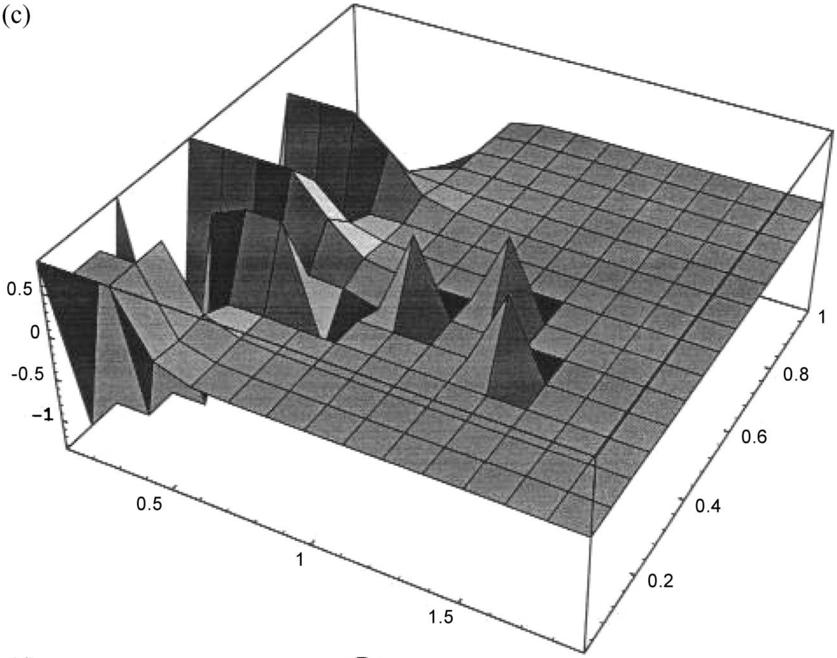


Fig. 5. The renormalised effective energy–momentum tensor for a SU(3) field. Figures (a) and (b) show the real respectively imaginary parts of $\langle T_{00}\rangle^{\text{Casimir}}$ for $r \in [0.2\text{M}, 8\text{M}]$ and $g \in [0.01, 10]$. Figures (c)–(f) zoom at the behaviour for small coupling constants, $g \in [0.01, 1.0]$. Figures (c) and (d) show the real and imaginary parts of T_{00} within Schwarzschild radius, $r/M \in [0.1, 1.9]$. Figures (e) and (f) show the real and imaginary parts of T_{00} outside the Schwarzschild radius, $r/M \in [2.1, 6.0]$.

(c)



(d)

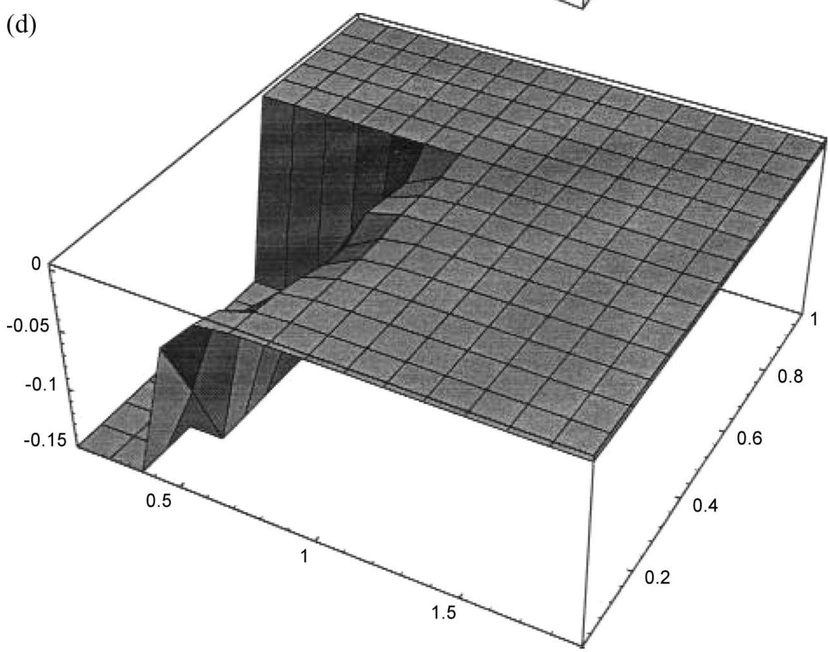
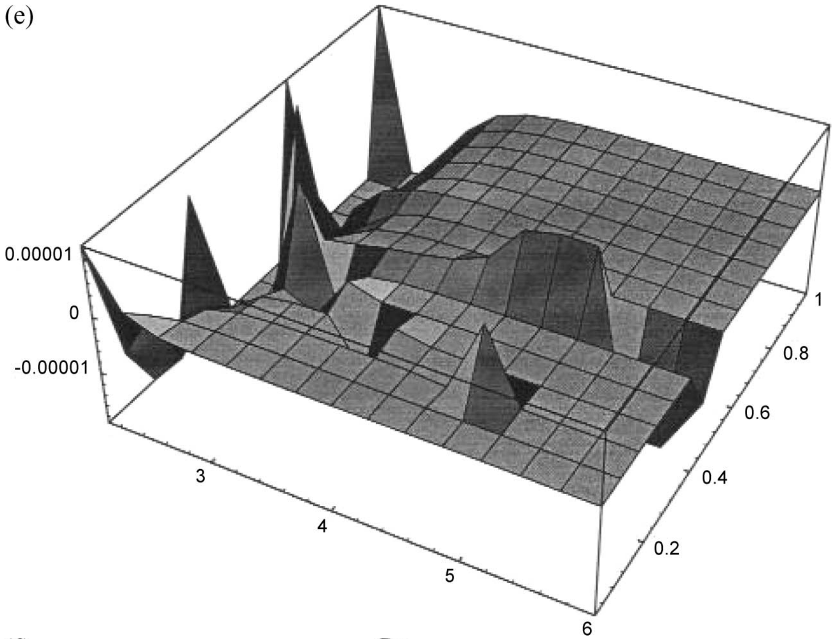


Fig. 5. Continued.

(e)



(f)

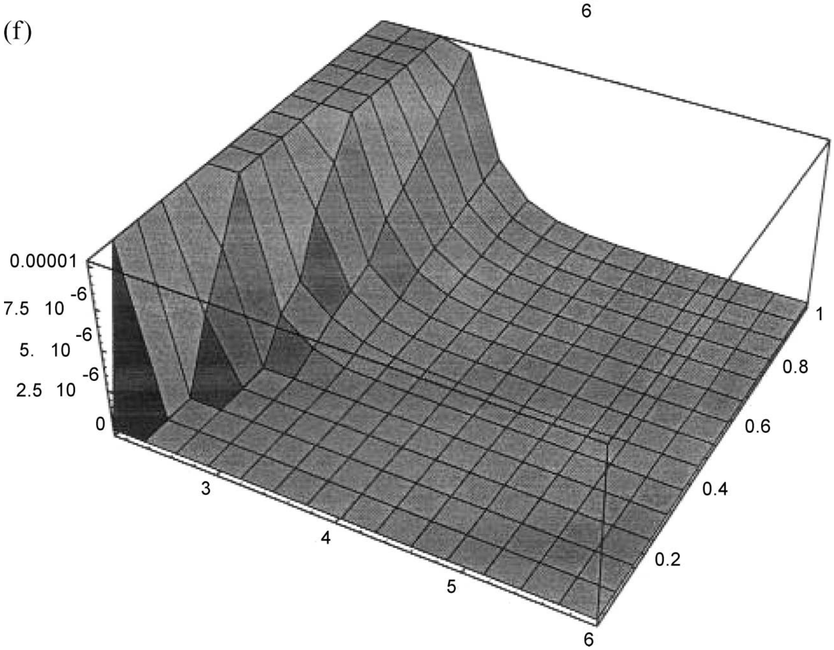


Fig. 5. Continued

higher and higher order (as one includes more terms in the truncated series (50)), leaving open the possibility that such resonances would totally dominate a fuller picture than that explored in this paper, perhaps with profound consequences to primordial black hole signatures.

In regard to the spectral signatures of primordial black holes it also should be noted that, comparing the size of the imaginary part of the energy–momentum tensor of fermions and scalars with that of the gauge field, one is lead to expect that the contribution from gauge bosons will be dominant although, in a full model, the coupling between fermions and gauge bosons probably would somewhat lessen the difference.

Discussing phenomenology it also deserves to be mentioned that one has a rather large radial pressure (the size of which is determined by what fields inhabits space–time) that might be of importance to the dynamics of the very early universe when matter is, in a sense, within its own Schwarzschild radius.

5. CONCLUSION

In this paper, effective Lagrangians as well as the part of the energy–momentum tensor stemming from the zero-point fluctuations of the quantum matter fields were calculated for scalar fields, spin 1/2 fermions and spin 1 gauge bosons, all quantum fields being free fields residing in a Schwarzschild geometry. In the case of the spin 0 and the spin 1/2 fields, the calculations were very clean and simple, the only approximation entering through the truncation of the series (18), renormalisation being done against a local Minkowski background. For gauge bosons, the calculations are essentially carried out along the same lines, however, the explicit introduction of the vierbeins (local, freely falling coordinate frames) introduces angular dependency of end results. This angular dependancy is due to the truncation of the series (50) and is unlikely to have any deeper significance. Furthermore, the angular-dependent contributions are minor ones and so the results one get from throwing them away can still be seen as order of magnitude indicators of the results one would get from a full calculation. And the results of the gauge boson calculation are interesting indeed, showing the interplay of the gravitational field and the gauge boson field (as signified by the gauge coupling constant) as regards particle creation (the imaginary part of the effective Lagrangian) and also showing that, especially the real part of the (zero-point energy contribution to the) energy–momentum tensor was dominated by 'resonances,' again signifying the interplay between gravitational and gauge boson field strengths.

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